Wave propagation

Remco Hartkamp (University of Twente)
Stefan Luding (University of Twente)
Katia Bertoldi (Harvard University)
Orion Mouraille (ASML)
Introduction

• Multiscale mechanics:
  Micro -> Meso -> Macro
Waves

- Through gas, liquid and solids
- Relatively small disturbances
Wave types

P: Pressure deviations
S: Shear stress deviations

Compressive (P) and Shear (S) waves

Sound:
20 Hz – 20 kHz

Gas: P
Liquid: P
Plasma: P
Solid: P & S

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Stretched string example

1D wave equation:

\[
\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}
\]

Harmonic solution of form:

\[
\phi(x, t) = A \exp\left\{i(kx - \omega t)\right\}
\]

\[\omega = ck, \text{ with } c \text{ independent of } k\]

However, in most problems.....

\[
\text{Wave number } k = \frac{2\pi}{\lambda}
\]

\[
\text{Angular frequency } \omega = 2\pi f
\]
Dispersion: Waves with different wavelengths propagate at different speeds

\[ c(k) = \frac{\omega(k)}{k} \]

Shallow water:

\[ c = \sqrt{gh} \]

Deep water:

\[ c = \sqrt{\frac{g\lambda}{2\pi}} \]
General recipe

1. Equations of motion:
   \[ \mathbf{M} \ddot{\mathbf{U}} + \mathbf{K} \mathbf{U} = 0 \]

2. Harmonic wave solution:
   \[ \mathbf{U} = \mathbf{U}_0 \cdot \exp \{ i (\mathbf{k} \cdot \mathbf{x} - \omega t) \} \]

3. Solve eigenvalue problem:
   \[ (\mathbf{K} - \omega (\mathbf{k})^2 \mathbf{M}) \mathbf{U}_0 = \mathbf{0} \]

- **M**: Mass-inertia matrix
- **K**: Stiffness matrix
- **U**: Displacement vector (translation & rotation)
- **U₀**: Eigenvector
- **ω²**: Eigenvalue
Application in granular materials

Earth:
- Complex
- Moist
- Polydisperse
- Stratified
P-wave in (dry) granular materials

- Monodisperse
- Regular lattice
- Periodic
- P-wave
- Ends fixed
- Frictionless
P-wave in (dry) granular materials

Frequency-space diagram

Dispersion relation

Color scale corresponds to the amplitude of Fourier coefficients.
large --> small

Grey-scale corresponds to the amplitude of the Fourier coefficients.
Waves in (dry) granular materials

Behavior can depend on:

- (dis)Order in structure
- Dispersity
- Contact model
- Friction
- Adhesion
- .....
Hard sphere vs. Soft sphere

Hard sphere: Repulsive (binary collisions)

Lennard Jones: Attractive + Repulsive

Mass-spring system: Attractive + Repulsive
1D mass spring system

2 modes:
- Acoustical vibration: particles in phase
- Optical vibration: particles in anti-phase

\[ m_1 = m_2 \]
\[ k_1 = k_2 \]
2D mass-spring system
Larger number of masses

.....towards continuum
Material model

Deformation gradient $\mathbf{F}$: $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$

Right Cauchy-Green tensor $\mathbf{C}$: $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$

Invariants:

$I_1 = \text{tr}(\mathbf{C})$

$I_2 = \frac{1}{2} \left( \text{tr}(\mathbf{C})^2 - \text{tr}(\mathbf{C}^2) \right)$

$I_3 = \text{det}(\mathbf{C})$

Strain energy density function: $W = W(I_1, I_2, I_3)$
Material model

First Piola-Kirchhoff stress tensor $\mathbf{P}$:

- Neo-Hookean model:
  
  $$
  \mathbf{P} = \mu \mathbf{F} + J \left[ K (J - 1) - \frac{\mu}{J} \right] \mathbf{F}^{-T}
  $$

- Gent model:
  
  $$
  \mathbf{P} = \frac{\mu J_m \mathbf{F}}{J_m - I_3 + 3} + J \left[ \left( \frac{K}{2} - \frac{2\mu}{J_m} \right)(J - 1) - \frac{\mu}{J} \right] \mathbf{F}^{-T}
  $$

$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}$

$J = \det(\mathbf{F})$

$\mu$ shear modulus

$K$ bulk modulus

$J_m$ material parameter (related to the strain saturation of the material)
Example, compression

\[ \mu = 0.21 \text{MPa} \]

\[ K = 1 \text{MPa} \]
FEM (Abaqus)

- Quadratic, triangular mesh
- Mode analysis
- Periodic unit cell
- Square array of circular holes

Irreducible Brillouin zone
Applications: sound filters, acoustic wave guides and acoustic mirrors.

Material and geometry to manipulate band gap.
• Finite structure
• Steady-State Dynamics (SSD) or dynamic simulation

Dynamic simulation
Color code = vertical displacement
Transfer (B/A)

Band gap

1

2

MSM

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Application of a simple load to a periodic structure can trigger an unexpected global pattern switch above a critical point. The results of numerical investigations reveal that the pattern switch is triggered by a reversible elastic instability.
Analysis of instabilities
(Bloch wave Analysis)

Bertoldi and Boyce, PRB, 2008

-Mathematical expressions
-Graphs and diagrams
-Experimental data

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The end

Thank you for your attention!